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On the Effect of Artificial-Viscosity Methods in Calculating Shocks

C. B. VREUGDENHIL

Delft Hydraulics Laboratory, Emmeloord, the Netherlands

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SUMMARY

Two representative difference-methods of first and second order accuracy are investigated as to their effect in computing shocks. It is shown that the effect of diffusing the shocks is caused entirely by the truncation-error. This opens the possibility of influencing the computed shock-thickness by suitably choosing the parameters in the difference-equations.

1. Introduction

Shock-like phenomena occur in a wide range of fields. In hydrodynamics some well-known examples are bores and hydraulic jumps. A less familiar example is the front of a sand-wave travelling along the bottom of a river [3]. This paper does not enter into the question whether these phenomena from a physical point of view can be described satisfactorily by shocks.

One way of calculating shocks is the application of a dissipative finite-difference scheme or artificial-viscosity method (Richtmyer and Morton [2]). By such a method shocks are spread over a certain distance and they sometimes are accompanied by secondary waves. Both effects may be a nuisance if they are pronounced. It would be useful to be able to influence the intensity of the smoothing effects. Therefore an investigation of the cause of these effects seems to be of some general interest. In this paper an idea is elaborated which is implicit in Richtmyer and Morton's treatment of artificial-viscosity methods and which considers the influence of the truncation-error. It turns out that this error accounts for a major part of the smoothing effect.

In order to simplify the analysis, a single linear differential equation is considered. As no shocks will produce spontaneously, one is provided for in the initial condition. The model problem is

$$u_t + cu_x = 0 \tag{1.1}$$

$$u(x,0) = \begin{cases} 1 & x \leqslant 0 \\ 0 & x > 0 \end{cases} \tag{1.2}$$

where t = time
x = spatial coordinate
u = dependent variable
c = velocity of propagation.

Some of the results will also be valid for non-linear problems by considering c to be variable with u. Qualitatively, they will also apply to systems of quasi-linear first-order equations, as these can be transformed into a form like eqn. (1.1).

2. Difference-Equations

A three-point explicit difference-scheme is considered, using mesh-widths Δt and Δx.

$$u_k^{n+1} = a_{-1} u_{k-1}^n + a_0 u_k^n + a_1 u_{k+1}^n \tag{2.1}$$

where u_k^n stands for $u(k\Delta x, n\Delta t)$. By expanding eqn. (2.1) into a Taylor-series with respect to the point (k, n), one finds

$$u+\Delta t u_t+\tfrac{1}{2}\Delta t^2 u_{tt}+\tfrac{1}{6}\Delta t^3 u_{ttt}+\ldots=$$
$$=(a_{-1}+a_0+a_1)u+\Delta x(a_1-a_{-1})u_x+\tfrac{1}{2}\Delta x^2(a_1+a_{-1})u_{xx}+\ldots \quad (2.2)$$

It follows that first order accuracy is achieved if

$$a_{-1}+a_0+a_1=1 \quad (2.3a)$$
$$-a_{-1} \quad +a_1=-\mu \quad (2.3b)$$

where $\mu=c\Delta t/\Delta x$. Equations (2.3) leave one degree of freedom. For $a_0=0$ Lax' scheme results (Lax [1]). From the differential equation $u_{tt}=c^2 u_{xx}$, so eqn. (2.2) becomes to first order

$$u_t+cu_x=\lambda_2 u_{xx}+\ldots \quad (2.4)$$

where $\lambda_2=(\Delta x^2/2\Delta t)\,(1-a_0-\mu^2)$. The right-hand side term is called the artificial viscosity-term. It vanishes if in addition

$$a_0=1-\mu^2\,. \quad (2.3c)$$

Then a scheme with second-order accuracy results which is equivalent to the Lax–Wendroff scheme [2]. In this case eqn. (2.2) to third order reads

$$u_t+cu_x=-\lambda_3 u_{xxx}+\lambda_4 u_{xxxx}+\ \ldots \quad (2.5)$$

where $\lambda_3=\dfrac{\Delta x^3}{6\Delta t}\mu(1-\mu^2)$

$$\lambda_4=\frac{\Delta x^4}{24\Delta t}\mu^2(1-\mu^2)\,.$$

Attention is called to the fact that the coefficient of artificial viscosity in eqn. (2.4) is positive only if

$$\mu^2\leqslant 1-a_0\,. \quad (2.6)$$

The same criterion is obtained by applying the Von Neumann-condition [2], which in this case is a necessary and sufficient condition for stability. A similar remark applies to the second-order scheme, where the Von Neumann-condition reads $\mu^2\leqslant 1$.

3. Interpretation

As a check on the diffusive effect of the truncation-error terms in eqns. (2.4) and (2.5), they can be integrated analytically by means of the Fourier-transform. A realistic initial condition is

$$u(x,0)=\begin{cases}1 & x\leqslant 0\\ 1-(x/\Delta x) & 0<x\leqslant\Delta x\\ 0 & x\geqslant\Delta x\end{cases} \quad (3.1)$$

which necessarily is used in numerical computations. The difference between (3.1) and (1.2) has a noticeable effect in the case of the second-order scheme. The solution of the first-order equation 2.4 is

$$u_1(x,t)=\tfrac{1}{2}+(\pi\Delta X_1)^{-1}\int_0^\infty\{\cos X_1 z-\cos(X_1-\Delta X_1)z\}z^{-2}e^{-z^2}dz \quad (3.2)$$

where $X_1=(x-ct)(\lambda_2 t)^{-\frac{1}{2}}$ and $\Delta X_1=\Delta x(\lambda_2 t)^{-\frac{1}{2}}$. The solution of the second-order equation (2.5) is

$$u_2(x,t)=\tfrac{1}{2}+(\pi\Delta X_2)^{-1}\int_0^\infty\{\cos(X_2 z+z^3)-\cos((X_2-\Delta X_2)z+z^3)\}\,z^{-2}e^{-Rz^4}dz \quad (3.3)$$

where $X_2=(x-ct)(\lambda_3 t)^{-\frac{1}{3}}$, $\Delta X_2=\Delta x(\lambda_3 t)^{-\frac{1}{3}}$ and $R=\lambda_4\lambda_3^{-1}(\lambda_3 t)^{-\frac{1}{3}}$.

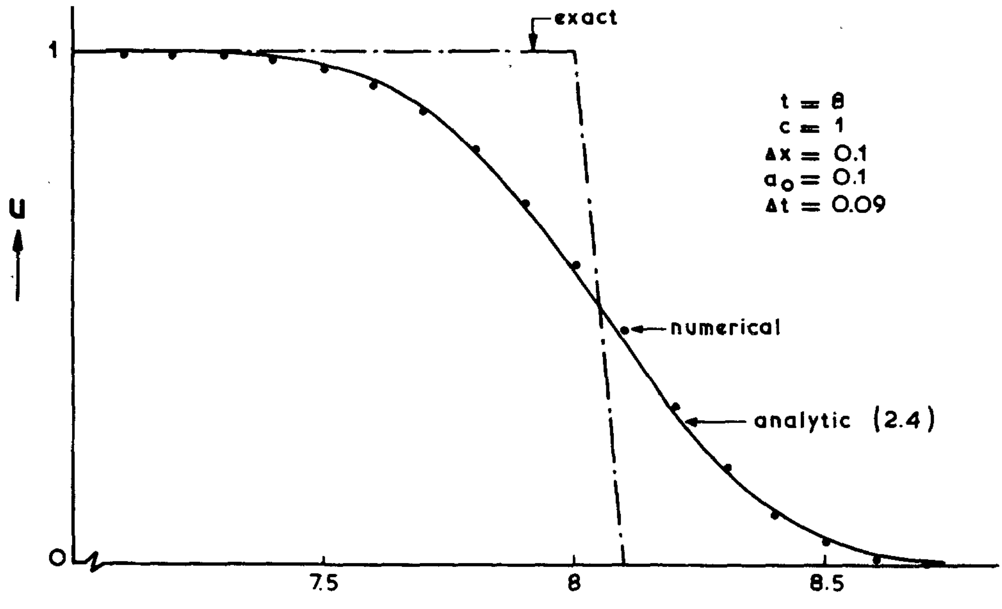

FIG. 1

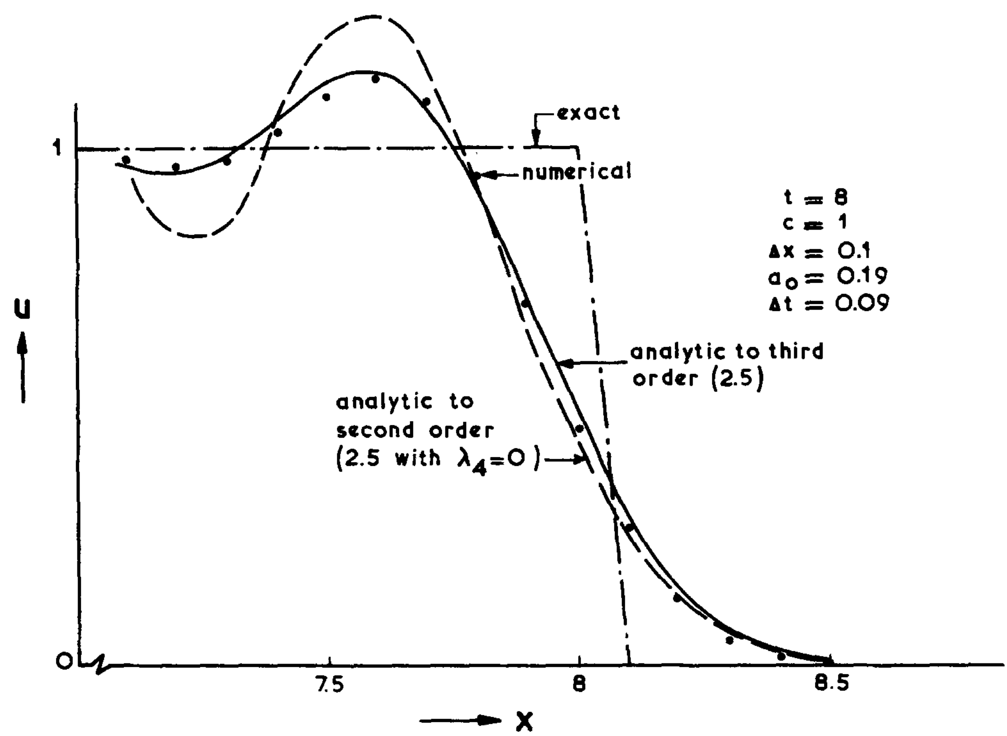

FIG. 2

In Figs. 1 and 2 these solutions are compared to the actual numerical ones. For the first-order scheme (Fig. 1) a close agreement is found. In the second-order case, the secondary waves are not explained satisfactorily if the first term of the truncation-error only is included ($R=0$). To this end, inclusion of the second term is required. At any rate, the secondary waves appear to be inherent in a second-order method.

As a consequence of these results, it turns out that the apparent shock-thickness can be influenced by a suitable choice of the parameters, apart from reducing the mesh-width. Of course the second-order method produces a sharper shock than does the first order method. However, in some of the physical situations, referred to in section 1, the secondary waves are very undesirable. Then a first-order method may be used in which $1-a_0-\mu^2$ is reduced as far as is admissible for stability. Similarly, in more general second-order methods the shock-thickness and the height of the secondary waves can be influenced.

The above considerations apply to linear equations only. For non-linear equations the situation may be similar, though more complicated.

4. Conclusions

Although the present analysis lacks any generality, the following conclusions can be drawn.

(i) The "spreading" of shocks and the secondary waves can be explained as consequences of the truncation error.

(ii) Both effects can be influenced by adjusting possible free parameters in the difference-equation.

(iii) In non-linear cases the results should be used with caution.

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